AD-A135 314 CONVERGENCE OF QUADRATIC FORMS IN P-STABLE RANDOM VARIABLES AND THETA SUB. (U) NORTH CAROLINA UNIV AT CHAPEL HILL DEPT OF STATISTICS S CAMBANIS ET AL. 1/1

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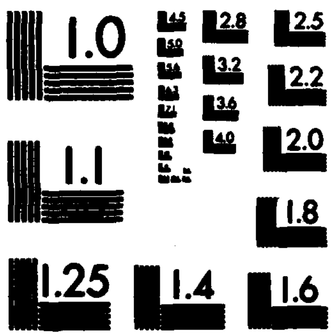

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CONVERGENCE OF QUADRATIC FORMS IN p-STABLE RANDOM VARIABLES AND θ_p-RADONIFYING OPERATORS

by

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and
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and
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TECHNICAL REPORT #41

October 1983

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 83-0975	2. GOVT ACCESSION NO. AD-A135314	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) CONVERGENCE OF QUADRATIC FORMS IN p-STABLE RANDOM VARIABLES AND θ_p-RADONIFYING OPERATORS		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL
		6. PERFORMING ORG. REPORT NUMBER TR #41
7. AUTHOR(s) Stamatis Cambanis and Jan Rosinski and W. Woyczynski		8. CONTRACT OR GRANT NUMBER(s) F49620-82-C-0009
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics University of North Carolina Chapel Hill NC 27514		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS PE61102F; 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS Mathematical & Information Sciences Directorate Air Force Office of Scientific Research /NM Bolling AFB DC 20332		12. REPORT DATE OCT 83
		13. NUMBER OF PAGES 20
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

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Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Quadratic forms, p-stable random variables, double Wiener integrals, θ_p-Radonifying operators.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Necessary and sufficient conditions are given for the almost sure convergence of the quadratic form $\sum\sum f_{jk} M_j M_k$ where (M_j) is a sequence of i.i d. p-stable random variables. A connection is established between the convergence of the quadratic form and a radonifying property of the infinite matrix operator (f_{kj}).

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CONVERGENCE OF QUADRATIC FORMS IN p-STABLE RANDOM VARIABLES AND θ_p-RADONIFYING OPERATORS

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(M sub j)

Summary

Necessary and sufficient conditions are given for the almost sure convergence of ~~the~~ a quadratic form $\sum\sum f_{jk}M_jM_k$ where $\cancel{(M_j)}$ is a sequence of i.i.d. p-stable random variables. A connection is established between the convergence of the quadratic form and a radonifying property of the infinite matrix operator $\cancel{(f_{kj})}$.

(f sub kj).

[1]Supported in part by the AFOSR Contract No. F49620 82 C 0009. The last named author would like to express his gratitude to the Center for Stochastic Processes of the University of North Carolina for its hospitality while this research was being carried out.

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I. INTRODUCTION

The aim of this paper is to study the convergence of the random quadratic forms of the form

$$(1.1) \qquad \sum_k \sum_j f_{jk} M_j M_k$$

where (f_{jk}), $j,k=1,2,3,\ldots$, is a real infinite matrix and (M_j), $j=1,2,\ldots$, is a sequence of i.i.d. p-stable random variables with characteristic function $\exp(-|t|^p)$, $0 < p < 2$. Our results have obvious implications in the theory of double Wiener-type integrals of the form

$$\iint f(x,y) M(dx) M(dy)$$

where $M(x)$ is a p-stable motion (cf. Szulga and Woyczynski (1983)). We shall study them elsewhere.

We begin with a characterization of non-anticipating sequences (V_k) such that the "martingale" transform

$$T = \sum V_k M_k$$

converges almost surely. The necessary and sufficient condition here turns out to be $\sum |V_k|^p < \infty$ a.s., and moreover T converges a.s. exactly on the set $(\sum |V_k|^p < \infty)$ (Theorem 2.1). This applied to the sequence

$$V_k = \sum_{j=1}^{k-1} f_{jk} M_j$$

shows that the convergence of the off-diagonal part of the iterated sum

$$\sum_k \Big(\sum_{j<k} f_{jk} M_j \Big) M_k$$

is equivalent to the matrix operator $(f_{jk})^T_{j<k}$ being θ_p-radonifying from ℓ^q into ℓ^p (Theorems 2.2 and 3.1).

Finally we characterize the class of θ_p-radonifying operators $(f_{jk})^T$ from ℓ^q into ℓ^p as satisfying the condition

$$\sum_k \{(\sum_j |f_{jk}|^p)(1+|\log \sum_j |f_{jk}|^p|)\} < \infty$$

(Corollary 2.2). The proof of necessity of this condition was shown to us by Gilles Pisier and is included here with his permission.

The above results permit the full characterization of all matrices (f_{jk}) for which the quadratic form (1.1) converges a.s. (Theorem 4.2). We would also like to mention here that the infinite quadratic forms satisfy a fairly general 0 - 1 law (cf. de Acosta (1976)).

Our results should be compared with the case where the M_j's are i.i.d. Gaussian, i.e. when $p = 2$. In this case Varberg (1966) has shown that the convergence of the series $\sum_k f_{kk}$ and $\sum_{jk} f_{jk}^2$ is necessary and sufficient for the quadratic mean convergence of (1.1), and the above conditions imply a.s. convergence of (1.1) as well. Moreover Lemma 3.7 of Rosinski and Szulga (1983) in conjunction with Kahana's (1968) inequality shows that the conditions are also necessary for a.s. convergence of the Gaussian quadratic form.

In Section 5 we include some auxiliary results and a conjecture of independent interest.

2. MARTINGALF TRANSFORMS AND CONVERGENCE OF STABLE TRIANGULAR QUADRATIC FORMS

THEOREM 2.1. *Let* $V_k = V_k(M_1, \ldots, M_{k-1})$, $k=2,3,\ldots$, *be a non-anticipating sequence of random variables. Then almost surely*

$$\{\sum V_k M_k \text{ converges}\} = \{\sum |V_k|^p < \infty\} .$$

Proof: Let $F_k = \sigma(M_1, \ldots, M_k)$, and

$$G(x) = P\{|M_1| > x\} \sim cx^{-p} ,$$

as $x \to \infty$, $c > 0$. Also let us set

$$A = \{\sum V_k M_k \text{ converges}\},$$

$$B = \{\sum |V_k|^p < \infty\}$$

and

$$C_n = \{|V_n M_n| > 1\} , \quad n=2,3,\ldots .$$

Then we have that

$$A \subset \{C_n \text{ i.o.}\}^c = \{\sum P(C_n | F_{n-1}) < \infty\} \text{ a.s.}$$

where the equality is implied by a conditional form of a Borel-Cantelli lemma (cf. Breiman (1968), p. 96). Since

$$P(C_n | F_{n-1}) = G(1/|V_n|) ,$$

we get that $G(|V_n|^{-1}) \to 0$ a.s. on A as $n \to \infty$. Therefore, for sufficiently large $N = N(\omega)$,

$$\sum_{n=N}^{\infty} |V_n|^p \le \frac{2}{c} \sum_{n=N}^{\infty} G(|V_n|^{-1}) < \infty$$

a.s. on A. This shows that $P(A \backslash B) = 0$.

Now we shall prove that $P(B \backslash A) = 0$. Let $\pi_n = \pi_n(\cdot, \omega)$ be a regular conditional distribution of $V_n M_n$ given F_{n-1}. We have that

$$\pi_n(E,\omega) = P(V_n M_n \in E | M_1, \ldots, M_{n-1}) = \mu_p(V_n^{-1}(\omega) E)$$

where μ_p is the distribution of M_1. By a theorem of Hill (1982), $\sum V_n(\omega) M_n(\omega)$ converges for almost all ω's for which the series $\sum X_n^{(\omega)}$ converges in probability

P', where $\{X_n^{(\omega)}(\omega')\}$ is a sequence of independent r.v.'s (defined on another probability space (Ω',P')) with distributions $\{\pi_n(\cdot,\omega)\}$. In our case the $X_n^{(\omega)}$'s have characteristic functions $\exp(-|V_n(\omega)|^p t^p)$. Thus $P(B\setminus A) = 0$ which completes the proof of the theorem. □

Theorem 2.1. enables us to translate the problem of a.s. convergence of quadratic forms into a more tractable problem of the a.s. convergence of series of independent random vectors with values in ℓ^p, whose standard basis is denoted by (e_k), $k=1,2,\ldots$.

THEOREM 2.2. *Let* $(f_{jk}: 1 \le j \le k-1,\ k \ge 2)$ *be a triangular matrix of real numbers, and* $x_j \overset{\text{def}}{=} (0,\ldots,\ 0, f_{j,j+1}, f_{j,j+2},\ldots) = \sum_{k=j+1}^{\infty} f_{jk} e_k$, $j=1,2,\ldots$. *Then*

$$\sum_{k=2}^{\infty} \Big(\sum_{j=1}^{k-1} f_{jk} M_j \Big) M_k$$

converges a.s. if and only if for every $j=1,2,\ldots$, $x_j \in \ell^p$ *and the vector random series* $\sum x_j M_j$ *converges a.s. in* ℓ^p.

<u>Proof</u>: Setting $V_k = \sum_{j=1}^{k-1} f_{jk} M_j$, $k=2,3,\ldots$, $V_1 = 0$, by Theorem 2.1 we have that the series $\sum V_k M_k$ converges a.s. if and only if $\sum |V_k|^p < \infty$ a.s., which is equivalent to the a.s. convergence in ℓ^p of the series $\sum e_k V_k$

Now assume that $S = \sum e_k V_k$ converges a.s. in ℓ^p. Since

$$\sum_{k=1}^{n+1} e_k V_k = \sum_{j=1}^{n} \Big(\sum_{k=j+1}^{n+1} f_{jk} e_k \Big) M_j ,$$

Proposition 5.1 applied to $x_{jn} = \sum_{k=j+1}^{n+1} f_{jk} e_k$, and $Y_j = M_j$, gives that $x_{jn} \to x_j = \sum_{k=j+1}^{\infty} f_{jk} e_k \in \ell^p$ as $n \to \infty$, and that the series $\sum x_j M_j$ converges a.s. in ℓ^p.

Conversely, assume that $x_j = \sum_{k=j+1}^{\infty} f_{jk} e_k \in \ell^p$ and $\sum x_j M_j$ converges a.s. in ℓ^p. The operator

$$\ell^p \ni x \to R_n(x) = \sum_{k=n+2}^{\infty} \langle x\ , e_k \rangle e_k \in \ell^p\ ,\ p > 0\ ,$$

is continuous and linear and $R_n(x) \to 0$ in ℓ^p for every x as $n \to \infty$. Thus a.s.

$$0 \underset{n}{\leftarrow} R_n\Big(\sum_{j=1}^{\infty} x_j M_j\Big) = \sum_{j=1}^{\infty} R_n(x_j)M_j = \sum_{j=1}^{n}\Big(\sum_{k=n+2}^{\infty} f_{jk}e_k\Big)M_j + \sum_{j=n+1}^{\infty} x_j M_j \ .$$

Hence,

$$\sum_{j=1}^{n}\Big(\sum_{k=n+2}^{\infty} f_{jk}e_k\Big)M_j \to 0$$

a.s. in ℓ^p as $n \to \infty$, and thus

$$\sum_{k=1}^{n+1} V_k e_k = \sum_{j=1}^{n} x_j M_j - \sum_{j=1}^{n}\Big(\sum_{k=n+2}^{\infty} f_{jk}e_k\Big)M_j$$

converges a.s. in ℓ^p. □

3. CONVERGENCE OF p-STABLE RANDOM SERIES IN ℓ^p AND θ_p-RADONIFYING OPERATORS

Definition 3.1. Let $0 < p < 2$ and F be a continuous linear operator $F: s \to \ell^p$, where s is the space of sequences with finite support for $0 < p \le 1$ and $s = \ell^q$, $1/q + 1/p = 1$, for $1 < p < 2$. We shall write $F = (f_{jk})$ where $f_{jk} = \langle Fe_j, e_k\rangle$, $j,k=1,2,\ldots$, so that $Fe_j = \sum_k f_{jk}e_k$. We say that F is in the class $\ell^p\log\ell(\ell^p)$ if for each $j=1,2,\ldots$, $Fe_j \in \ell^p$ and the ℓ^p-valued sequence (Fe_j) is in $\ell^p\log\ell$. In other words, $F \in \ell^p\log\ell(\ell^p)$ if and only if

$$N(F) = \sum_j \|Fe_j\|_p^p(1 + |\log\|Fe_j\|_p^p|)$$

$$= \sum_j \{(\sum_k |f_{jk}|^p)(1 + |\log \sum_k |f_{jk}|^p|)\} < \infty .$$

One can introduce on $\ell^p\log\ell(\ell^p)$ an Orlicz type norm in the natural fashion.

THEOREM 3.1. *Let* $0 < p < 2$ *and* $x_j = \sum_{k=1}^{\infty} f_{jk}e_k \in \ell^p$, $j=1,2,\ldots$. *Then* $\sum x_jM_j$ *converges a.s. in* ℓ^p *if and only if* $F^T = (f_{jk})^T \in \ell^p\log\ell(\ell^p)$.

Proof: Sufficiency. Assume $F^T \in \ell_p\log\ell(\ell_p)$. We shall prove that the assumptions of Proposition 5.2 are fulfilled for the series with general term $W_k = |\sum_j f_{jk}M_j|^p$, which has the same distribution as $(\sum_j|f_{jk}|^p)|M_1|^p$. Indeed,

$$\sum_k P(W_k > 1) = \sum_k P(|M_1| > (\sum_j |f_{jk}|^p)^{-1/p}) \le \text{Const} \quad \sum_{jk}\sum |f_{jk}|^p < \infty$$

since, by assumption, $\sum_k\sum_j|f_{jk}|^p < \infty$ and $\sum_j|f_{jk}|^p \to 0$ as $k \to \infty$. Therefore (5.1) of Proposition 5.2 is satisfied.

Now,

$$B = \sum_k E\{W_kI(W_k \le 1)\} = \sum_k(\sum_j|f_{jk}|^p)E\{|M_1|^pI(|M_1| \le (\sum_j|f_{jk}|^p)^{-1/p})\}.$$

Denoting the distribution function of $|M_1|$ by Φ, we get that for $\alpha > 1$

$$\int_0^{\alpha}|u|^pd\Phi(u) \le 1 + \alpha^p\Phi(\alpha) - \Phi(1) - \int_1^{\alpha}pu^{p-1}\Phi(u)du$$

$$\le 1+\alpha^p\Phi(\alpha) - \Phi(1) - \int_1^\alpha pu^{p-1}(1-(C/u^p))\,du$$

since $\Phi(u) = 1 - P(|M_1| > u) \ge 1 - Cu^{-p}$ for $u \ge 1$ and certain constant C. Thus because $\alpha^p(\Phi(\alpha) - 1)$ remains bounded

$$\int_0^\alpha |u|^p d\Phi(u) \le \text{Const} \quad (1+\log\alpha).$$

It follows that

$$B \le \text{Const} \sum_k \sum_j |f_{jk}|^p (1 + |\log \sum_j |f_{jk}|^p|) < \infty$$

so that Condition (5.2) of Proposition 5.2 is also satisfied and as a result $\sum_k|\sum_j f_{jk}M_j|^p < \infty$ a.s. It follows that $\sum_k(\sum_j f_{jk}M_j)e_k$ converges a.s. in ℓ^p and in view of $x_j = \sum_k f_{jk}e_k \in \ell^p$, we have that $\sum_j x_jM_j$ converges a.s. in ℓ^p.

Necessity. (The basic idea here is due to Pisier). Assume that $\sum x_jM_j = \sum (Fe_j)M_j$ converges in ℓ_p a.s. By the representation for stable processes obtained in Proposition 1.5 of Marcus and Pisier (1983) and used here in the discrete paramater case (in the case of $1\le p<2$, Corollary 3 of LePage, Woodroofe and Zinn (1981) would suffice) $\sum x_jM_j$ converges in ℓ^p a.s. if and only if $\sum \varepsilon_jY_j\Gamma_j^{-1/p}$ converges a.s. in ℓ^p, where (ε_j) are independent Bernoulli r.v.'s, (Y_j) are i.i.d. r. elements in ℓ^p with law

$$L(Y_j) = \text{Const} \sum_k \|x_k\|^p \delta_{x_k/\|x_k\|},\ j=1,2,\ldots,$$

and $\Gamma_j = X_1 + \ldots + X_k$ where $X_1, X_2\ldots$ are i.i.d. with $P(X_1 > u) = e^{-u}$. Moreover the sequences $(\varepsilon_j),(Y_j),(\Gamma_j)$ are independent of each other. By the law of large numbers, $\Gamma_j/j \to 1$ a.s. Applying twice Fubini's theorem and the comparison principle in quasi-normed spaces obtained in Theorem 4.4 of Marcus and Woyczynski (1979), we get that the series $\sum x_jM_j$ converges in ℓ^p a.s. if and only if $\sum \varepsilon_jY_jj^{-1/p}$ does. What is useful about the latter series is that it possesses all moments in contrast to the former.

Therefore for $0 < p < 2$, and $x_j = Fe_j$, $j=1,2,\ldots$,

$$\infty > E\|\sum_j j^{-1/p}\varepsilon_j Y_j\|_p^p = \sum_k E|\sum_j j^{-1/p}\varepsilon_j Y_{jk}|^p$$

$$\text{(2.2)} \qquad \geq \text{Const} \sum_k E(\sum_j j^{-2/p} Y_{jk}^2)^{p/2}$$

$$\geq \text{Const} \sum_k E(\sup_j j^{-1/p}|Y_{jk}|)^p$$

by Khintchine's inequality ($Y_j = (Y_{j1}, Y_{j2}, \ldots)$).

If C denotes the class of all stopping times, then for all k,

$$E|\sup_j j^{-1/p} Y_{jk}|^p \geq \sup_{\tau\in C} E\tau^{-1}|Y_{\tau k}|^p \geq \text{Const} \quad E(|Y_{1k}|^p \log|Y_{1k}|)$$

by Chow, Robbins & Siegmund (1971), p. 97 , where the last constant can be chosen greater than 0 (uniformly in k), since in our case

$$\mathcal{L}(Y_{jk}) = \text{Const} \sum_i \|Fe_i\|_p^p \, \delta_{<Fe_i, e_k>\|Fe_i\|_p^p} \, .$$

Therefore

$$E(|Y_{1k})|^p \log|Y_{1k}|) = \sum_i \frac{|f_{ik}|^p}{\|Fe_i\|_p^p} \log \frac{|f_{ik}|^1}{\|Fe_i\|_p^p} \cdot \|Fe_i\|_p^p$$

so that finally we get from (2.2) that

$$\sum_k \sum_j |f_{jk}|^p \log \frac{|f_{jk}|}{\sum_k |f_{jk}|^p} < \infty$$

which certainly implies that $F^T \in \ell^p \log \ell(\ell^p)$. □

COROLLARY 3.1. *Let* $\sum_k |f_{jk}|^p < \infty$. *Then* $\sum_j |\sum_k f_{jk} M_k|^p < \infty$ *a.s. if and only if* $\sum_j \{(\sum_k |f_{jk}|^p)(1 + |\log \sum_k |f_{jk}|^p|)\} < \infty$.

Now let $1 < p \leq 2$ and let θ_p be a cylindrical measure on ℓ^q, $1/p + 1/q = 1$, generated by the sequence $M = (M_i)$, $i=1,2,\ldots$. The characteristic functional of θ_p is given by the formula

$$\int_{\ell^q} e^{i<x,y>} \theta_p(dy) = e^{-\|x\|_p^p}, \quad x \in \ell^p \, .$$

Definition 3.2. *Let* E *be a Banach space. A linear operator* $F:\ell^q \to E$ *is called* θ_p*-radonifying if* $\exp(-\|F^*y^*\|_p^p)$, $y^* \in E^*$, *is the characteristic functional of a Radon measure on* E. *We denote by* $R_p(\ell^q,E)$ *the class of all such operators.*

The class $R_2(\ell^2,E)$ has been extensively studied. The main result is that $\Pi_2(\ell^2,E) = R_2(\ell^2,E)$ if and only if E is of cotype 2 (cf. Linde and Pietsch (1974)), where Π_p denotes the ideal of all p-absolutely summing operators. For $1 < p < 2$ it has been proved that if $1 < r < p$ then $\Pi_r(\ell^q,E) \subset R_p(\ell^q,E)$, and that $\Pi_p(\ell^q,E) \subset R_p(\ell^q,E)$ if and only if E is isomorphic to subspace of L^p and is of stable type p (cf. Linde, Mandrekar, Weron (1980), and Thang and Tien (1982)).

From the now classical results due to Itô and Nisio (1968) it is easy to deduce the following.

THEOREM 3.2. *Let* $1 < p \leq 2$. *The following are equivalent:*

(a) $F \in R_p(\ell^q,E)$;

(b) $F^*:E^* \to \ell^p$ *is decomposable in the sense that there exists an* E*-valued, strongly measurable random vector* (FM) *such that*

$$\langle F^*y^*,M\rangle = \langle y^*,FM\rangle \quad \text{a.s.}, \; y^* \in E^*;$$

(c) The series $\sum F(e_j)M_j$ *converges a.s. in* E, *where* (e_j) *is the standard basis in* ℓ^q.

The above equivalences are heuristically better understood if one keeps in mind the following "identities":

$$Ee^{i\langle T^*y^*,M\rangle} = Ee^{i\sum_k M_k\langle T^*y^*,e_k\rangle} = e^{-\sum_k|\langle T^*y^*,e_k\rangle|^p} = e^{-\|T^*y^*\|_p^p}.$$

In view of the above remarks every r-absolutely summing F, $1 < r < p$, is θ_p-radonifying, but there are p-absolutely summing $F:\ell^q \to \ell^p$ which are not θ_p-radonifying since ℓ^p is not of stable type p. The corollary below, obtained immediately from Theorems 3.1 and 3.2, reflects this fact and gives an analytic description of θ_p-

radonifying operators F from $\ell^q \to \ell^p$.

COROLLARY 3.2. $[R_p(\ell^q,\ell^p)]^T = \ell^p \log \ell(\ell^p)$.

4. CONVERGENCE OF GENERAL QUADRATIC FORMS IN p-STABLE RANDOM VARIABLES

Let

$$Q_n = \sum_{k=1}^{n} \sum_{j=1}^{n} f_{jk} M_j M_k, \quad n=1,2,\ldots,$$

where (M_j), $j=1,2,\ldots$, are as above i.e. i.i.d. with $E\exp(itM_j) = \exp(-|t|^p)$, $0 < p < 2$. Denote the diagonal and off-diagonal parts of Q_n respectively by

$$D_n = \sum_{k=1}^{n} f_{kk} M_k^2 \quad \text{and} \quad R_n = \sum_{\substack{k,j=1 \\ k \neq j}}^{n} f_{jk} M_j M_k .$$

The diagonal part, being a series of independent random variables, is easy to handle.

THEOREM 4.1. *The sequence* (D_n) *converges a.s. as* $n \to \infty$ *if and only if*

$$\sum_{k=1}^{\infty} |f_{kk}|^{p/2} < \infty.$$

Proof. Let us observe that $\sum P(|f_{kk}| M_k^2 > 1) < \infty$ if and only if $\sum |f_{kk}|^{p/2} < \infty$. Kolmogorov's three series theorem gives now the "only if" part of the claim. The proof of the "if" part follows directly from Proposition 5.2 because for $p<2$ and $f_{kk} \neq 0$,

$$E|f_{kk}| M_k^2 I(|f_{kk}| M_k^2 \leq 1) = \int_0^1 P(M_k^2 > t|f_{kk}^{-1}|)dt \leq c|f_{kk}|^{p/2} \int_0^1 t^{-p/2} dt = \text{Const } |f_{kk}|^{p/2}. \ \square$$

The above result and the results of previous sections give the following.

THEOREM 4.2. *The sequence of quadratic forms* (Q_n) *converges a.s. as* $n \to \infty$ *if and only if*

$$\sum_{k=1}^{\infty} |f_{kk}|^{p/2} < \infty$$

and

$$\sum_{k=2}^{\infty} \{(\sum_{j=1}^{k-1} |f_{jk} + f_{kj}|^p)(1 + |\log \sum_{j=1}^{k-1} |f_{jk} + f_{kj}|^p|) < \infty .$$

Proof: The above result is a straightforward corollary to Theorems 4.1, 2.2, and 3.1 and to the fact that the sequence (Q_n) converges a.s. if and only if both sequences (D_n) and (R_n) converge a.s. We prove the only if part of this latter assertion.

Let a,b be reals with $a \neq 0$, and set $\varepsilon = \text{sign}(b/a)$. Then for $M = M_1$, we have

$$
\begin{aligned}
P\{|aM^2 + bM| > 1\} &= P\{|a(\varepsilon M)^2 + b(\varepsilon M)| > 1\} = P\{||a|M^2 + |b|M| > 1\} \\
&\geq P\{||a|M^2 + |b|M| > 1,\ M > 0\} \\
&= P\{|a|M^2 + |b|M > 1\ ,\ M > 0\} \\
&\geq P\{M > |a|^{-1/2}\} = \frac{1}{2} P\{|M| > |a|^{-1/2}\}\ .
\end{aligned}
\tag{4.1}
$$

Assume that (Q_n) converges a.s. We have $Q_n = \sum_{k=1}^n (V_k + f_{kk}M_k)M_k$ where $V_k = \sum_{j=1}^{k-1}(f_{jk} + f_{kj})M_j$, $k \geq 2$, $V_1 = 0$. By the conditional Borel-Cantelli lemma (cf. Breiman (1968), p. 96), the a.s. convergence of (Q_n) implies that

$$\sum_{k=1}^{\infty} P\{|V_kM_k + f_{kk}M_k^2| > 1 | M_1, \ldots, M_{k-1}\} < \infty \text{ a.s.}$$

It follows by (4.1) that $\sum P\{|f_{kk}M_k^2| > 1\} < \infty$, where the sum extends over all k for which $f_{kk} \neq 0$, which implies that $\sum_{k=1}^{\infty} |f_{kk}|^{p/2} < \infty$, in view of the tail behavior of the M_k's. Now, by Theorem 4.1, D_n converges a.s. and by the assumption $R_n = Q_n - D_n$ converges a.s. as well. □

In view of Theorem 5.1 below and of Theorem 2.1, we have the following corollary on the convergence in L^r, $r < p$, of the off-diagonal part of the quadratic form.

COROLLARY 4.1. *If* $0 < r < p$ *and*

$$\sum_{k=2}^{\infty} \Big(\sum_{j=1}^{k-1} |f_{jk} + f_{kj}|^p\Big)^{r/p} < \infty$$

then the sequence $R_n = \sum_{k \neq j}^n f_{jk}M_jM_k$ *converges in* L^r *and a.s.*

Proof: Here is a direct proof which does not use Theorems 2.1 and 5.1. By the martingale property of R_n we have, with $V_k = \sum_{j=1}^{k-1}(f_{jk} + f_{kj})M_j$, $k \geq 2$, $V_1 = 0$,

$$E|R_n - R_{m+1}|^r = E|\sum_{k=m}^{n} V_k M_k|^r \leq \text{Const} \sum_{k=m}^{n} E|V_k M_k|^r$$

$$= \text{Const} \sum_{k=m}^{n} E|V_k|^r E|M_k|^r$$

$$= \text{Const} \sum_{k=m}^{n} \left(\sum_{j=1}^{k-1} |f_{jk} + f_{kj}|^p\right)^{r/p}$$

which proves the corollary. □

5. AUXILIARY RESULTS

The proof of Theorem 2.2 relies on the following technical property which, roughly speaking, justifies the change of the order of summation in the series $\sum_j(\sum_{k>j} f_{jk}e_k)M_j$.

PROPOSITION 5.1. *Let* E *be a complete metric linear space,* $x_{jk} \in E$, $n,j=1,2,\dots,$ *and* (Y_j) *a sequence of non-zero independent symmetric real random variables. If*

$$S_n = \sum_{j=1}^{n} x_{jn}Y_j \to S$$

in probability as $n \to \infty$ *then there exists a sequence* $(x_j) \subset E$ *such that for each* j, $x_{jn} \to x_j$ *as* $n \to \infty$ *and the series* $\sum x_jY_j$ *converges a.s. to* S.

Proof: Let $\|\cdot\|$ be a monotonic F-norm on E, i.e. $\|ax\| \le \|x\|$ for $|a| \le 1$ (which always exists by Rolewicz (1972), p. 16, Theorem I.2.2). Fix $j \ge 1$ and let $a,b > 0$ be such that $P\{|Y_j| > a\} > b$. Let $\varepsilon > 0$. For $r \ge n \ge j$ we set $c(r,n) = 1$ if $\| a(x_{jr} - x_{jn}) \| > \varepsilon$ and $c(p,n) = 0$ otherwise. Since for a symmetric pair of r. vectors X,Y in E,

$$P(\|X\| > \varepsilon) \le 2P(\| \tfrac{1}{2} (X + Y) \| > \tfrac{\varepsilon}{2})$$

we obtain

$$b\cdot c(r,n) \le P\{\| Y_j(x_{jr} - x_{jn}) \| > \varepsilon\} \le 2P\{\| \tfrac{1}{2} \sum_{k=1}^{n} Y_k(x_{kr} - x_{kn}) \| > \tfrac{\varepsilon}{2}\}$$

$$\le 4P\{\| \tfrac{1}{4} (S_r - S_n) \| > \tfrac{\varepsilon}{4}\} .$$

It then follows from the assumption that $c(r,n) \to 0$ as $r,n \to \infty$, i.e. $\{x_{jr}\}$ as a Cauchy sequence for every j, and by the completeness of E there exists $x_j \in E$ such that $x_{jn} \to x_j$ as $n \to \infty$.

Let now $\varepsilon > 0$ and let N be such that

$$P\{\| \tfrac{1}{2} (S_r - S_n) \| > \tfrac{\varepsilon}{2}\} \le \tfrac{\varepsilon}{2}$$

for every $r \ge n \ge N$. By the symmetry argument used above

$$P\{\|\sum_{j=1}^{n} Y_j(x_{jr} - x_{jn})\| \geq \varepsilon\} \leq \varepsilon \ .$$

Keeping n fixed and letting $r \to \infty$ we get

$$P\{\|\sum_{j=1}^{n} x_j Y_j - S_n\| \geq \varepsilon\} \leq \varepsilon \ .$$

Thus $\sum_{j=1}^{n} x_j Y_j \to S$ in probability and, since the Y's are independent, also a.s. □

The following elementary proposition (cf. Szulga and Woyczynski (1983)) is used in the proof of Theorem 3.1.

PROPOSITION 5.2. *If for a sequence* (W_k) *of real random variables*

(5.1) $\qquad \sum_k P(|W_k| > 1) < \infty$,

and

(5.2) $\qquad \sum_k E|W_k I(|W_k| \leq 1)| < \infty$

then $\sum |W_k| < \infty$ *a.s.*

<u>Proof</u>: Indeed, let $Y_k = W_k I(|W_k| \leq 1)$, $Z_k = W_k - Y_k$. Then $|W_k| = |Y_k| + |Z_k|$. $\sum |Y_k|$ converges a.s. since $E(\sum |Y_k|) < \infty$ by (5.2), and $\sum |Z_n|$ converges a.s. by the Borel-Cantelli lemma. □

We now establish a more precise criterion for summability of stable r.v.'s which is used in Section 4. For a symmetric p-stable r.v. X the quantity c_X is defined by $E\exp(itX) = \exp(-c_X|t|^p)$ and satisfies

$$\|X\|_{L^r} = (E|X|^r)^{1/r} = \mathrm{Const}(r,p)\, c_X^{1/p} \quad , \ 0 < r < p,$$

and for independent symmetric p-stable r.v.'s (X_k),

(5.3) $\qquad c_{\sum_k a_k X_k} = \sum_k |a_k|^p c_{X_k}$.

<u>Definition 5.1</u>. *The r.v.'s (X_k) are jointly symmetric p-stable if for every sequence (a_k) with a finite number of nonzero elements the r.v. $\sum_k a_k X_k$ is symmetric p-stable.*

THEOREM 5.1. *Let (X_k) be jointly symmetric p-stable r.v.'s with $0 < p < 2$ and let $r > 0$. Then a necessary and sufficient condition for*

$$\sum_{k=1}^{\infty} |X_k|^r < \infty \text{ a.s.}$$

is that for some $0 < s < p$,

$$E\Big(\sum_{k=1}^{\infty} |X_k|^r\Big)^s < \infty \quad \text{when } 0 < r \le 1$$

and

$$E\Big(\sum_{k=1}^{\infty} |X_k|^r\Big)^{s/r} < \infty \quad \text{when } 1 < r \; .$$

<u>Proof</u>: Assume $X = (X_k)_k \in \ell^r$ a.s. and define $\Phi:\Omega \to \ell^r$ by $\Phi(\omega) = (X_k(\omega))_k = X(\omega)$ if $(X_k(\omega))_k \in \ell^r$ and $\Phi(\omega) = 0$ otherwise. Then Φ induces a symmetric p-stable measure $\mu = P\circ\Phi^{-1}$ on ℓ^r. For $x = (x_k)_k \in \ell^r$ define $q(x) = \sum_k |x_k|^r$ when $0 < r \le 1$ and $q(x) = (\sum_k |x_k|^r)^{1/r}$ when $r > 1$. Then q is a measurable seminorm on ℓ^r (a norm when $r \ge 1$) and by Theorem 3.2 in de Acosta (1975) we have for $0 < s < p$,

$$E\{q(X(\omega))\}^s = \int_\Omega q^s(X(\omega))\,dP(\omega) = \int_{\ell^r} q^s(x)\,d\mu(x) < \infty \; .$$

The converse is clear. □

When $1 < r < p < 2$ we can take $s = r$ and the necessary and sufficient condition becomes

$$\sum_{k=1}^{\infty} E|X_k|^r < \infty \quad \text{or} \quad \sum_{k=1}^{\infty} c_{X_k}^{r/p} < \infty \; .$$

When $r = p$, Theorem 5.1 gives

(5.4) $$\sum_{k=1}^{\infty} |X_k|^p < \infty \text{ a.s. if and only if } E\Big(\sum_{k=1}^{\infty} |X_k|^p\Big)^{s(1\wedge\frac{1}{p})} < \infty \text{ for some } 0 < s < p.$$

This necessary and sufficient condition simplifies to

(5.5) $$\sum_{k=1}^{\infty} c_{X_k}(1 + |\log c_{X_k}|) < \infty$$

when the X_k's are independent, by Schwartz's theorem (cf. e.g. Woyczynski (1978), p. 277), and when the X_k's are of the form $X_k = \sum_j f_{jk}M_j$, where the M_j's are independent (cf. Corollary 3.1 and (5.3)). Since every sequence of jointly symmetric p-stable r.v.'s (X_k) is of the form $X_k = \int_0^1 f_k(t)dM(t)$, $k=1,2,\ldots$, where $M(t)$, $0 \le t \le 1$, is a stable motion (i.e. has independent stationary symmetric p-stable increments) and $\int_0^1 |f_k(t)|^p dt < \infty$, $k=1,2,\ldots$ (cf. Kuelbs (1973)), we conjecture that (5.5) is always a necessary and sufficient condition for (5.4).

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